Fuzzy Logics and Games

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(joint work with Petr Cintula)

LoMoReVI 2009

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We shall call this procedure a *negotiation game*.

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So they play bargaining/negotiation games.

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$x \oplus y$	$\min(1, x + y)$	strong disjunction
$x \vee y$	$\max(x, y)$	weak disjunction
$x \wedge y$	$\min(x, y)$	weak-conjunction
$\neg x$	$1 - x$	negation
$x \otimes y$	$\max(0, x + y - 1)$	strong conjunction

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(0) $(\varphi, v, 0)$: the end of the game,
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$\mathcal{F}$ chooses whether to play (ψ_1, v, r') or $(\psi_2, v, r - r')$.

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Conjunction

Evaluation games

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Evaluation games

Negation

($\neg$) $(\neg\psi, v, r)$:
$\mathcal{F}$ chooses $r', r \geq r' > 0$
role switch, game continues as $(\psi, v, (1 - r) + r')$

Evaluation games

Existential quantifier
Verifier's choice of an element from the domain of the model witnessing the claim
$||(\exists x)\psi||_v \geq r$, which is equivalent to $\sup(||\psi||_{v[x]}) \geq r$

($\exists'$) $((\exists x)\psi, v, r)$:
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$\mathcal{V}$ chooses $a \in M$
the game continues as $(\psi, v[x : a], r)$.

this works if the supremum is witnessed,
if it is proper, i.e., $||\psi||_{v[x:a']} < \sup(||\psi||_{v[x]})$ for all $a' \in M$, $\mathcal{V}$ never finds a witness.

Evaluation games

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($\exists$) $((\exists x)\psi, v, r)$:
$\mathcal{F}$ chooses $r' < r$ and $\mathcal{V}$ chooses $a \in M$,
the game continues as $(\psi, v[x : a], r')$.

Evaluation games

General quantifier
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General quantifier

The position $((\forall x)\psi, v, r)$ corresponds to Verifier's claim that $\inf(||\psi||_{v[x]}) \geq r$.

$\mathcal{F}$ is to move and he has to provide a counterexample, i.e., to find an a' such that $(||\psi||_{v[x:a']} < r)$.

In this case the (non)existence of the witnessing element does not influence Falsifier's choice, so the move consists just of a choice.

($\forall$) $((\forall x)\psi, v, r)$:
$\mathcal{F}$ chooses $a \in M$
game continues as $(\psi, v[x : a], r)$.

Correspondence theorem

Fuzzy evaluation games are zero-sum games of a finite depth, so by Zermelo's theorem they are determined. We can prove the correspondence between the existence of winning strategies in a fuzzy game and the standard Tarskian truth.

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Theorem

Let **L** *be an* MV*-chain,* **M** *be a safe* **L***-structure,* φ *a formula, v an* **M***-valuation, and r* $\in$ *L. Then Eloise has a winning strategy for the* $(\mathbf{M}, \mathbf{L})$*-Game* (φ, v, r) *iff* $||\varphi||^{\mathbf{L}}_{\mathbf{M},v} \geq r$*.*

Safe structures

The interpretation of $\exists, \forall$, the truth assignment in fuzzy logics is in general a *partial* function. To make it a total function we have to restrict the class of models in consideration.

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Definition

Let Γ be a predicate language, **Ł** an MV-chain, **M** an **Ł**-structure for Γ. We say that:

- **M** is a *safe* **Ł**-structure, if $||\varphi||^{\mathbf{Ł}}_{\mathbf{M},v}$ is defined for each φ and *v*.
- **M** is a *witnessed* **Ł**-structure, iff for each φ and *v* there are $a, b \in M$ such that $||(\exists x)\varphi||^{\mathbf{Ł}}_{\mathbf{M},v} = ||\varphi||^{\mathbf{Ł}}_{\mathbf{M},v[x:=a]}$ and $||(\forall x)\varphi||^{\mathbf{Ł}}_{\mathbf{M},v} = ||\varphi||^{\mathbf{Ł}}_{\mathbf{M},v[x:=b]}$.
  (Replace sup and inf by max and min.)

We restrict interpretation of formulas to safe structures only.

Winning strategies

We did not use the notion of a safe structure in our definition of evaluation game. Is the game interpretation more general?

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Definition

Let **L** be an MV-chain, **M** be an **L**-structure, φ a formula, and v an **M**-valuation. We define:
$\mathcal{WS}_v^+(\varphi) =_{\mathrm{df}} \{r \mid$ Eloise has a winning strategy for the game $G(\varphi, v, r)\}$.
$\mathcal{WS}_{\varphi,v}^-(\varphi) =_{\mathrm{df}} \{r \mid$ Abelard has for any $r' > r$ a winning strategy for the game $G(\varphi, v, r')\}$.

Winning strategies - properties

Lemma

Let **L** *be an* MV*-chain,* **M** *be an* **L***-structure,* φ *a formula, and v an* **M***-valuation. Then:*

1. $0 \in \mathcal{WS}^+(\varphi)$
2. $1 \in \mathcal{WS}^-(\varphi)$
3. $\mathcal{WS}^+(\varphi)$ *is a lower set*
4. $\mathcal{WS}^-(\varphi)$ *is an upper set;*
5. $\mathcal{WS}^-(\varphi) \cup \mathcal{WS}^+(\varphi) = L$;
6. $||\mathcal{WS}^-(\varphi) \cap \mathcal{WS}^+(\varphi)|| \leq 1$;
7. *For a* safe **M***:* $\mathcal{WS}^+(\mathbf{M}, \mathbf{L}, v, \varphi) \cap \mathcal{WS}^-(\mathbf{M}, \mathbf{L}, v, \varphi) = \{||\varphi||^{\mathbf{L}}_{\mathbf{M},v}\}$.

Winning strategies - properties

Lemma

Let **L** *be an* MV*-chain,* **M** *be an* **L***-structure,* φ *a formula, and* v *an* **M***-valuation. Then:*

1. $\mathcal{WS}^+(\varphi \oplus \psi) = \{r \oplus s \mid r \in \mathcal{WS}^+(\varphi) \text{ and } s \in \mathcal{WS}^+(\psi)\}$
2. $\mathcal{WS}^+(\varphi \otimes \psi) = \{r \otimes s \mid r \in \mathcal{WS}^+(\varphi) \text{ and } s \in \mathcal{WS}^+(\psi)\}$;
3. $\mathcal{WS}^+(\varphi \vee \psi) = \mathcal{WS}^+(\varphi) \cup \mathcal{WS}^+(\psi)$;
4. $\mathcal{WS}^+(\varphi \wedge \psi) = \mathcal{WS}^+(\varphi) \cap \mathcal{WS}^+(\psi)$;
5. $\mathcal{WS}^+(\neg\varphi) = \{\neg r \mid r \in \mathcal{WS}^-(\varphi)\}$;
6. $\mathcal{WS}^+((\forall x)\varphi) = \bigcap_{a \in M} \mathcal{WS}^+(v[x = a], \varphi)$;
7. $\mathcal{WS}^+((\exists x)\varphi) = \mathcal{C}lo(\bigcup_{a \in M} \mathcal{WS}^+(v[x = a], \varphi))$;

Winning strategies and non-safe structures

Example

Let **L** be the subalgebra of the standard MV-algebra with the domain $[0,1] \cap Q$.
Let q be an irrational number grater then $\frac{1}{2}$ and let a_i be a sequence of rationals descending to q.
Let **M** be the **L**-structure of a predicate language with one unary predicate P, where the domain of **M** is the set of natural numbers and $P_{\mathbf{M}}(i) = a_i$.
M is not a safe **L**-structure–the truth value of $\forall x P(x)$ is undefined.
$\mathcal{WS}^+(\forall x P(x)) = [0, q)$.
The truth value of $\varphi = (\forall x)P(x) \oplus (\forall x)P(x)$ is also undefined.
However we know that, $\mathcal{WS}^+(\varphi) = [0,1]$ and $\mathcal{WS}^-(\varphi) = \{1\}$. Thus $\mathcal{WS}^+(\varphi) \cap \mathcal{WS}^-(\varphi) = \{1\}$.

G-tautology

Definition

Let **L** be an MV-chain and **M** be an *arbitrary* **L**-structure. Then: φ is a G-tautology iff $\mathcal{WS}_v^+(\varphi) = \mathbf{L}$ for any **M**-valuation v.

Lemma

Let **L** be an MV-chain and **M** be a *safe* **L**-structure. Then: φ is a G-tautology iff $(\mathbf{M}, \mathbf{L}) \models \varphi$

G-interpretation

Definition

Let **L** be an MV-chain, **M** be an *arbitrary* **L**-structure and *v* an **M**-valuation. We call the set $\mathcal{WS}_v^+(\varphi)$ the *G-interpretation* of the formula φ.

Lemma

Let **L** *be an* MV*-chain,* **M** *be an arbitrary* **L***-structure and v an* **M***-valuation.*

1. $G(\varphi \oplus \psi) = G(\varphi) \oplus G(\psi)$
2. $G(\varphi \otimes \psi) = G(\varphi) \otimes G(\psi)$
3. $G(\varphi \wedge \psi) = G(\varphi) \cap G(\psi)$
4. $G(\varphi \vee \psi) = G(\varphi) \cup G(\psi)$
5. $G(\exists \psi) = G(\varphi) \cup G(\psi)$
6. $G(\forall x)\varphi = \bigcap_{a \in M} G_{v[x=a]}\varphi$
7. $G(\exists x)\varphi = \mathcal{C}lo(\bigcup_{a \in M})G_{v[x=a]}\varphi$